Counterexamples in Algebraic Geometry

Jules Zhu

April 9, 2026

Contents

Abstract

This note collects some fundamental examples that illustrate subtle points in scheme theory and sheaf theory. Each example is presented in detail with the necessary background.

1 Sheaf surjectivity does not imply surjectivity on global sections

Let X be a topological space and let $\mathscr{F} \twoheadrightarrow \mathscr{G}$ be a surjective morphism of sheaves of abelian groups (or $\mathcal{O}_X$-modules). Surjectivity means that for every open $U \subseteq X$ and every section $s \in \mathscr{G}(U)$ there exists an open covering $U = \bigcup_i U_i$ and sections $t_i \in \mathscr{F}(U_i)$ such that $t_i \mapsto s|_{U_i}$. It does *not* guarantee that a global section $s \in \mathscr{G}(X)$ lifts to a global section of $\mathscr{F}$.

Example 1.1 (Exponential sequence)**.** *Let $X = \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ with the usual complex topology, and let $\mathcal{O}_X$ be the sheaf of holomorphic functions, $\mathcal{O}_X^\times$ the sheaf of nowhere vanishing holomorphic functions (both as sheaves of abelian groups under addition and multiplication respectively). The exponential map*

$$\exp : \mathcal{O}_X \longrightarrow \mathcal{O}_X^\times, \qquad f \mapsto e^{2\pi i f}$$

is a surjective morphism of sheaves: locally every nonvanishing holomorphic function admits a holomorphic logarithm. However, the induced map on global sections

$$\exp : \mathcal{O}_X(X) \longrightarrow \mathcal{O}_X^\times(X)$$

is not *surjective. The identity function $z \in \mathcal{O}_X^\times(X)$ has no global logarithm on $\mathbb{C}^\times$. Indeed, if $\log z$ were a holomorphic function on $\mathbb{C}^\times$, its derivative would be $1/z$, whose integral around the unit circle gives $2\pi i \neq 0$, contradicting the existence of a primitive.*

This example is the starting point for sheaf cohomology: the cokernel of $\exp$ *on global sections is precisely $H^1(X, \mathbb{Z})$.*

Remark 1.1. *In the algebraic category the same phenomenon occurs: on $X = \operatorname{Spec} k[t, t^{-1}]$, the Kummer sequence $0 \to \mu_n \to \mathbb{G}_m \xrightarrow{(\cdot)^n} \mathbb{G}_m \to 1$ is exact as a sequence of sheaves for the étale topology, but the map on global sections $k^\times \to k^\times$ may fail to be surjective if k does not contain enough roots of unity.*

2 A map of ringed spaces that is not a morphism of locally ringed spaces

A morphism of schemes is by definition a morphism of locally ringed spaces, i.e. a pair $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ where $f : X \to Y$ is continuous and $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a map of sheaves of rings such that for every

$x \in X$ the induced homomorphism on stalks $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a *local* homomorphism (it sends the maximal ideal into the maximal ideal). The next example shows a map of ringed spaces that satisfies all conditions except locality on stalks.

Example 2.1 (Spectrum of a discrete valuation ring)**.** *Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and fraction field K. Its spectrum $S = \operatorname{Spec} R$ consists of two points: the closed point $s = \mathfrak{m}$ and the generic point $\eta = (0)$. The scheme $T = \operatorname{Spec} K$ consists of a single point t. The canonical inclusion $R \hookrightarrow K$ induces a morphism of schemes*

$$\varphi : T \longrightarrow S, \qquad t \longmapsto \eta,$$

which is a morphism of locally ringed spaces (the stalk at t is K, the stalk at η is also K, and the map is the identity).

Now consider the set-theoretic *map*

$$\psi : T \longrightarrow S, \qquad t \longmapsto s.$$

We can equip ψ with a morphism of sheaves of rings: since $\mathcal{O}_T(T) = K$ and $\psi_*\mathcal{O}_T(T) = \mathcal{O}_T(\psi^{-1}(T)) = \mathcal{O}_T(t) = K$, we define $\psi^\sharp$ on the global sections as the composition

$$\mathcal{O}_S(S) = R \hookrightarrow K \xrightarrow{\text{id}} K = (\psi_*\mathcal{O}_T)(S).$$

This defines a morphism of sheaves because S has a basis of open sets $\{S, \{\eta\}\}$ and one checks compatibility. However, the induced map on stalks at t is

$$\psi_t^\sharp : \mathcal{O}_{S,s} = R_\mathfrak{m} \longrightarrow \mathcal{O}_{T,t} = K,$$

which is again the inclusion $R \hookrightarrow K$. This map is not local*: the maximal ideal $\mathfrak{m}R_\mathfrak{m}$ of $R_\mathfrak{m}$ is sent to a non-unit in K (in fact to 0 if we consider the residue field, but K is a field so the only maximal ideal is (0)). Hence $(\psi, \psi^\sharp)$ is a map of ringed spaces but not a morphism of locally ringed spaces.*

Remark 2.1. *This example underscores why the "local homomorphism" condition is essential in the definition of a scheme morphism: it forces the continuous map to send a point x to the unique point y whose local ring maps to the local ring of x in a local way, which for affine schemes corresponds exactly to the contraction of prime ideals along a ring homomorphism.*

3 Non-affine schemes obtained by gluing

A scheme is called *affine* if it is isomorphic to the spectrum of a commutative ring. Gluing affine schemes along open subsets often produces schemes that are no longer affine. Two classical examples follow.

3.1 The affine line with doubled origin

Let k be a field. Consider two copies of the affine line:

$$X_1 = \operatorname{Spec} k[x], \qquad X_2 = \operatorname{Spec} k[y].$$

Let $U_1 = X_1 \setminus \{(x)\} = D(x) = \operatorname{Spec} k[x]_x = \operatorname{Spec} k[x, x^{-1}]$[1]. Similarly, $U_2 = X_2 \setminus \{(y)\} = \operatorname{Spec} k[y, y^{-1}]$. We can glue X_1 and X_2 along U_1 and U_2 via the identity isomorphism or the inverse one. We first try the identity gluing,

$$\phi : U_1 \xrightarrow{\sim} U_2, \qquad x \longmapsto y.$$

The resulting scheme X is called the *affine line with doubled origin*. Its underlying topological space is the affine line with two copies of the origin, each having its own local ring $\mathcal{O}_{X,0_i} \cong k[x]_{(x)}$.

Example 3.1 (Non-affineness of the doubled origin line). *The scheme X is* not *affine. One way to see this is to compute its ring of global sections. The structure sheaf satisfies*

$$\mathcal{O}_X(X) = \ker\bigl(\mathcal{O}_{X_1}(X_1) \times \mathcal{O}_{X_2}(X_2) \rightrightarrows \mathcal{O}_{U_1}(U_1)\bigr),$$

where the two arrows are restriction and restriction composed with ϕ, $\approx$ kernel of sending (f, g) to $(f - g)$. Explicitly, as in Example 2.3.5 in Hartshorne,

$$\mathcal{O}_X(X) = \{(f(x), g(y)) \in k[x] \times k[y] \mid f(x) = g(x) \text{ in } k[x, x^{-1}]\}.$$

Since polynomials are determined by their restriction to the punctured line[2]*, this forces $f = g$ as elements of $k[x]$. Hence $\mathcal{O}_X(X) \cong k[x]$. If X were affine, by Proposition 2.2 in Hartshorne* [3]*, it would be isomorphic to* $\operatorname{Spec} k[x]$*, but the latter has a single origin $\{(x)\}$, while X has two distinct points $\{(x), (y)\}$ whose local rings are isomorphic to $k[x]_{(x)}$ or $k[y]_{(y)}$. Contradiction.*

[1] Here $D(x)$ is exactly the image of $\operatorname{Spec} k[x]_x \to \operatorname{Spec} k[x]$ induced by $i : k[x] \to k[x]_x$.
[2] Remark that $k[x] \to k[x]_x$ is injective.
[3] $\mathcal{O}(D(f)) \cong A_f$, in particular $\Gamma(\operatorname{Spec} A, \mathcal{O}) \cong A$ when $f = 1$.

3.2 The projective line

Using the same notation, glue X_1 and X_2 via the isomorphism

$$\psi : U_1 \xrightarrow{\sim} U_2, \qquad x \longmapsto y^{-1}.$$

The resulting scheme is by definition the projective line $\mathbb{P}^1_k$. It is covered by two affine charts $D_+(x_0) = \{[x_0 : x_1] \mid x_0 \neq 0\} \cong \{\frac{x_1}{x_0} \in k\} \cong \mathbb{A}^1_k \cong \operatorname{Spec} k[x]$ and $D_+(x_1) = \{[x_0 : x_1] \mid x_1 \neq 0\} \cong \{\frac{x_0}{x_1} \in k\} \cong \mathbb{A}^1_k \cong \operatorname{Spec} k[y]$ which play the roles of X_1, X_2 with $\psi : x = \frac{x_1}{x_0} \mapsto (\frac{x_0}{x_1})^{-1} = y^{-1}$ in our setup.

Example 3.2 ($\mathbb{P}^1_k$ is not affine)**.** *Although $\mathbb{P}^1_k$ is separated (in fact a variety), it is not affine. Its ring of global sections is*

$$\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = \ker\big(k[x] \times k[y] \rightrightarrows k[x, x^{-1}]\big),$$

where the arrows send $f(x)$ to $f(x)$ and $g(y)$ to $g(x^{-1})$. A pair $(f(x), g(y))$ lies in the kernel iff $f(x) = g(x^{-1})$ as elements of $k[x, x^{-1}]$. Expanding $g(y) = \sum_{i=0}^m a_i y^i$, we get $g(x^{-1}) = \sum a_i x^{-i}$, which can equal a polynomial $f(x)$ only if all $a_i = 0$ for $i > 0$ and $a_0 = f(0)$. Thus $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) \cong k$. If $\mathbb{P}^1$ were affine, it would be isomorphic to Spec *k, a single point, which is absurd.*

Remark 3.1. *Both examples illustrate that gluing affine schemes can produce non-affine schemes. The doubled origin line fails to be separated (the two distinct origins cannot be separated by disjoint open sets), while $\mathbb{P}^1$ is separated but not affine. Together they show that the category of schemes is strictly larger than the category of affine schemes.*

Further reading

These examples are standard in introductory texts on algebraic geometry:

- R. Hartshorne, *Algebraic Geometry*, Chapter II.