How to consider Hurewicz Theorem from a perspective of Natural Transformation?

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April 1, 2026

1 Introduction

Recall that the reduced homology groups satisfy $\widetilde{H}_n(S^n) \cong \mathbb{Z}$. Fix generators $i_n \in \widetilde{H}_n(S^n)$ that are compatible with the isomorphisms

$$\widetilde{H}_n(S^n) \cong H_n(D^n, S^{n-1}) \cong \widetilde{H}_{n-1}(S^{n-1}).$$

Definition 1 (Hurewicz homomorphism)**.** *For laziness, write $X = (X, x_0)$ a pointed topological space. For each $n \geq 1$ we define the* ***Hurewicz homomorphism (or Hurewicz map)***

$$h_n : \pi_n(X) \longrightarrow H_n(X)$$

as follows. Given a pointed map $f : S^n \to X$ representing a class $[f] \in \pi_n(X)$, the induced homomorphism $f_* : \widetilde{H}_n(S^n) \to \widetilde{H}_n(X)$ (or $f_* : H_n(S^n) \to H_n(X)$ in absolute homology) sends the generator i_n to some element. Since $\widetilde{H}_n(X) \cong H_n(X)$ for $n \geq 1$ when X is non-empty, we set

$$h_n([f]) = f_*(i_n) \in H_n(X).$$

This assignment is ***well-defined*** *because homotopic maps induce the same homology homomorphism.*

Theorem 1 (Hurewicz Theorem)**.** *For $n = 1$ the Hurewicz map factors through the abelianization $\pi_1(X)^{\mathrm{ab}}$ and gives an isomorphism $\pi_1(X)^{\mathrm{ab}} \cong H_1(X)$; for $n \geq 2$, if X is $(n-1)$-connected (i.e., $\pi_i(X) = 0$ for $i < n$), then h_n is an isomorphism.*

Remark that if X is any topological space, then $\pi_n(X, x_0)$ is abelian for $n \geq 2$ where x_0 is any base point. See HERE for more informations.

Definition 2 (Natural Transformation)**.** *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories and let $F, G : \mathbf{C} \to \mathbf{D}$ be two functors. A* ***natural transformation*** $\eta : F \Rightarrow G$

consists of a family of morphisms $\eta_X : F(X) \to G(X)$ *in* $\mathbf{D}$*, one for each object* X *of* $\mathbf{C}$*, such that for every morphism* $f : X \to Y$ *in* $\mathbf{C}$ *the diagram commutes:*

$$\begin{CD} F(X) @>{F(f)}>> F(Y) \\ @V{\eta_X}VV @VV{\eta_Y}V \\ G(X) @>>{G(f)}> G(Y) \end{CD}$$

2 The Hurewicz Homomorphism as a Natural Transformation

For each $n \geq 1$ we have two functors from the category $\mathbf{Top}_*$ of pointed topological spaces to the category $\mathbf{Ab}$ of abelian groups:

- $\pi_n : \mathbf{Top}_* \to \mathbf{Ab}$ is the n-th homotopy group functor.
- $H_n : \mathbf{Top}_* \to \mathbf{Ab}$ is the n-th singular homology group functor.

(For $n = 1$ we implicitly compose with abelianization, since π_1 is not abelian in general, but the Hurewicz map factors through π_1^{ab}; one can view π_1^{ab} as a functor to $\mathbf{Ab}$.)

The Hurewicz homomorphisms $h_n^X : \pi_n(X) \to H_n(X)$ assemble into a natural transformation

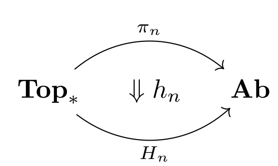
$$h_n : \pi_n \Longrightarrow H_n.$$

Indeed, for any pointed map $f : X \to Y$, the following square commutes:

$$\begin{CD} \pi_n(X) @>{\pi_n(f)}>> \pi_n(Y) \\ @V{h_n^X}VV @VV{h_n^Y}V \\ H_n(X) @>>{H_n(f)}> H_n(Y) \end{CD}$$

This is precisely the naturality condition, which follows from the functoriality of homology: $H_n(f)(h_n^X([g])) = (f \circ g)_*([S^n]) = h_n^Y([f \circ g]) = h_n^Y(\pi_n(f)([g]))$.

A schematic overview of the natural transformation h_n is given by the following diagram in the functor category:



with a double arrow $\Downarrow h_n$ indicating the natural transformation. More concretely, for each object $X \in \mathbf{Top}_*$ we have a morphism $h_n^X : \pi_n(X) \to H_n(X)$, and these are compatible with morphisms in $\mathbf{Top}_*$ as shown in the commutative square above.