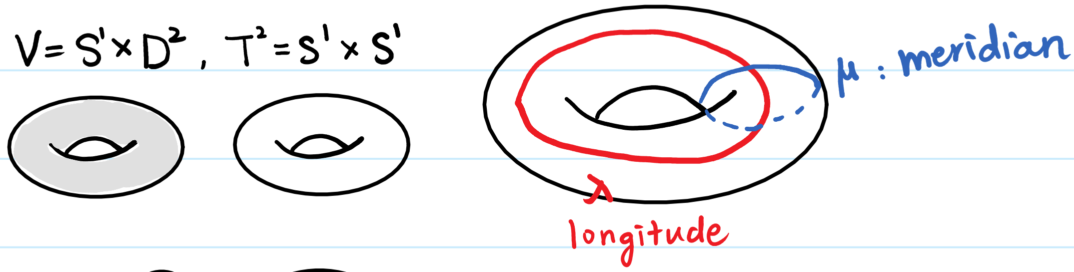
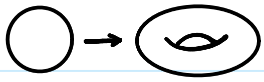
Prob 6.7 (lemma)

$S^1 \subset \mathbb{C}$, $V = S^1 \times D^2$, $T^2 = S^1 \times S^1$



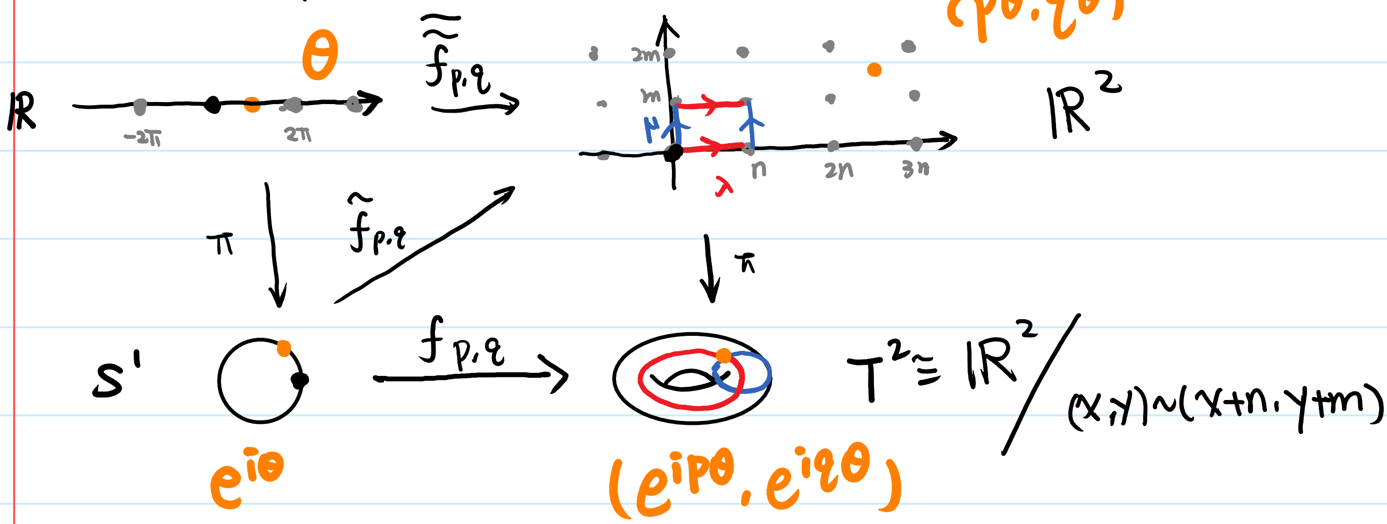
$p, q \in \mathbb{Z}$, $f_{p,q}$:  . Denote its image by $\gamma_{p,q}$.

$e^{i\theta} \longmapsto (e^{ip\theta}, e^{iq\theta})$

Given that $H_1(T^2) = \mathbb{Z}[\lambda] \oplus \mathbb{Z}[\mu]$, prove:

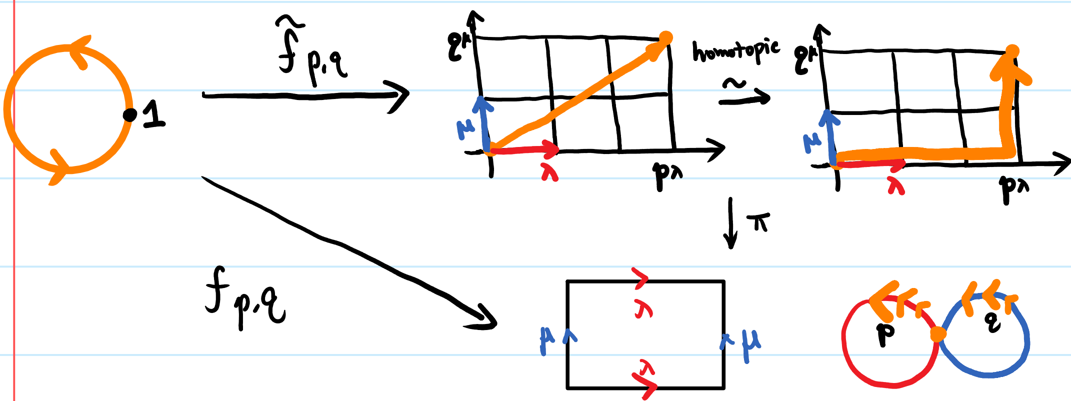
$$[\gamma_{p,q}] = p[\lambda] + q[\mu]$$

Pf: Lift $f_{p,q}$ to its universal cover:



$\tilde{\tilde{f}}_{p,q}\big|_{[0,2\pi)} : [0, 2\pi) \to \mathbb{R}^2$, shows that images of S^1 in $\mathbb{R}^2$ and T^2 are:

$\theta \longmapsto (p\theta, q\theta)$



In $\mathbb{R}^2$, hence in T^2, $r_{p,q}$ is homotopic to $p\lambda + q\mu$, so

$$\boxed{[\gamma_{p,q}] = p[\lambda] + q[\mu]}$$

□

6.8: Lens Space

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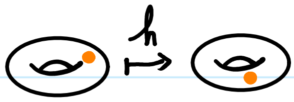

Prob 6.8 $(p,q)=1$, $\exists\, s,t$: $\det\begin{pmatrix} s & p \\ t & q\end{pmatrix}=1$. $p,q,s,t\in\mathbb{Z}$

V_1, V_2: solid tori with surfaces T_1^2, T_2^2. Define homeo $h: T_1^2 \xrightarrow{\sim} T_2^2$

$$(e^{i\theta}, e^{i\varphi}) \mapsto (e^{i\theta'}, e^{i\varphi'})$$

$$\begin{pmatrix}\theta' \\ \varphi'\end{pmatrix} = \begin{pmatrix} s & p \\ t & q\end{pmatrix}\begin{pmatrix}\theta \\ \varphi\end{pmatrix}$$

$$\Rightarrow \mathcal{L}(p,q) := V_1 \cup_h V_2.$$

Show $H_n(\mathcal{L}(p,q))$.

Pf: Since h is a homeomorphism, $\exists\, X_1, X_2 \subset \mathcal{L}(p,q)$ satisfy:

(1) $X_1 \cup X_2 = \mathcal{L}(p,q)$

(2) $X_1 \cong V_1$, $X_2 = V_2$, $X_1\cap X_2 = T_2^2$ (h glues V_1 onto V_2)

(3) $X_1 \cap X_2$ is a deformation retract of an open nbhd V.

(Consider $W_1 \overset{\subset X_1}{\cong} T_1^2 \times [0,\varepsilon)$, $W_2 \overset{\subset X_2}{\cong} T_2^2\times[0,\varepsilon)$, then

T_1^2, T_2^2 are retracts of W_1, W_2 relatively and W_1, W_2 are open in X_1, X_2 relatively.

$\Rightarrow W_1 \cup W_2 \cong T_2^2 \times (-\varepsilon, \varepsilon)$ is the V as desired.)

By Cor 4.12, (X_1, X_2) is a MV-pair of $\mathcal{L}(p,q)$, with

$$\begin{tikzcd} & \underset{(\sim S^1)}{X_1} \arrow[dr, "j_1"] & \\ \underset{(=T^2)}{X_1\cap X_2} \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & X_1\cup X_2 = \mathcal{L}(p,q) \\ & \underset{(\sim S^1)}{X_2} \arrow[ur, "j_2"'] & \end{tikzcd},$$

which induces 2 exact sequences:

(1) $$0 \to S^*(X_1\cap X_2) \xrightarrow{h_\#} S^*(X_1)\oplus S^*(X_2) \xrightarrow{k_\#} S^*(X_1)+S^*(X_2) \to 0$$

(2) $$\cdots \to \underset{=T_2^2}{\tilde{H}_q(X_1\cap X_2)} \xrightarrow{h_q} \underset{\sim S^1}{\tilde{H}_q(X_1)} \oplus \underset{\sim S^1}{\tilde{H}_q(X_2)} \xrightarrow{k_q} \underset{=\mathcal{L}(p,q)}{\tilde{H}_q(X_1\cup X_2)} \xrightarrow{\partial_q} H_{q-1}(X_1\cap X_2) \to \cdots$$

6.8 (2)

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Since when $q \geq 3$, $\tilde{H}_q(X_1 \cap X_2) = \tilde{H}_q(X_1) \oplus \tilde{H}_q(X_2) = 0$,
we only need to study $q = 3, 2, 1$.

(1). $\cdots \to \cancel{\tilde{H}_3(X_1\cap X_2)}^{0} \to \cancel{\tilde{H}_3(X_1)\oplus \tilde{H}_3(X_2)}^{0} \to \tilde{H}_3(X_1 \cup X_2) \xrightarrow{\sim} \tilde{H}_2(X_1\cap X_2) \to \cancel{\tilde{H}_2(X_1)\oplus \tilde{H}_2(X_2)}^{0} \to \cdots$

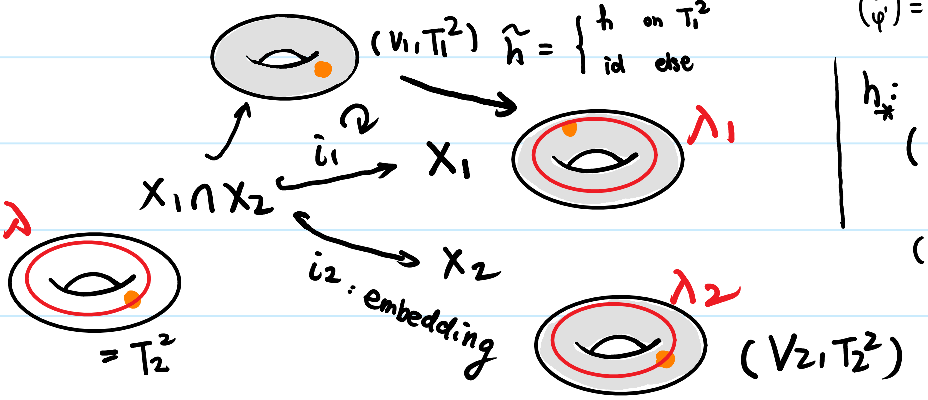
$\Rightarrow \tilde{H}_3(L(p,q)) \cong \tilde{H}_2(T_2^2) = \mathbb{Z}$

(2) $\cdots \to \cancel{\tilde{H}_2(X_1)\oplus \tilde{H}_2(X_2)}^{0} \to \underbrace{\tilde{H}_2(X_1\cup X_2)}_{L(p,q)} \xrightarrow{\partial_2} \underbrace{\tilde{H}_1(X_1\cap X_2)}_{=T_2^2} \xrightarrow{h_1} \underbrace{\tilde{H}_1(X_1)}_{\simeq S^1} \oplus \underbrace{\tilde{H}_1(X_2)}_{\simeq S^1} \xrightarrow{k_1} \underbrace{H_1(X_1\cup X_2)}_{=L(p,q)} \xrightarrow{\partial_1} \cancel{H_0(X_1\cap X_2)}^{0}$

i.e. $0 \to \tilde{H}_2(L(p,q)) \xrightarrow{\partial_2} \underbrace{\tilde{H}_1(X_1\cap X_2)}_{\mathbb{Z}[\lambda]\oplus\mathbb{Z}[\mu]} \xrightarrow{h_1} \underbrace{\tilde{H}_1(X_1)\oplus \tilde{H}_1(X_2)}_{\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{k_1} H_1(L(p,q)) \to 0$

$$\boxed{\begin{aligned} &\text{so } \tilde{H}_2(L(p,q)) \cong \operatorname{Im}\partial_2 = \ker h_1 \\ &\tilde{H}_1(L(p,q)) \cong \tilde{H}_1(X_1)\oplus\tilde{H}_1(X_2)/\ker(k_1) = \mathbb{Z}\oplus\mathbb{Z}/\operatorname{Im}(h_1). \end{aligned}} \quad --- \ (*)$$

The key is to find $h_1 : \tilde{H}_1(X_1\cap X_2) \to \tilde{H}_1(X_1) \oplus \tilde{H}_1(X_2)$ induced by



$\begin{pmatrix}\theta' \\ \varphi'\end{pmatrix} = \begin{pmatrix} s & p \\ t & q\end{pmatrix}\begin{pmatrix}\theta \\ \varphi\end{pmatrix}$

$h_*: [\lambda] \mapsto [\gamma_{s,t}]$
$(e^{i\theta}, e^{i0}) \mapsto (e^{is\theta}, e^{it\varphi})$
$[\mu] \mapsto [\gamma_{p,q}]$
$(e^{i0}, e^{i\varphi}) \mapsto (e^{ip\theta}, e^{iq\varphi})$

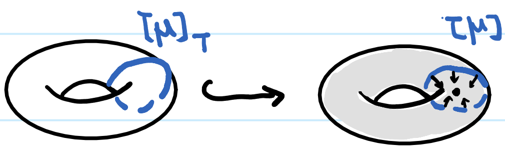
i.e.

$h_1 : \tilde{H}_1(X_1\cap X_2) \to \tilde{H}_1(X_1) \oplus \tilde{H}_1(X_2)$

$\mathbb{Z}[\lambda] + \mathbb{Z}[\mu] \qquad \mathbb{Z}\oplus\mathbb{Z}$

$[\lambda] \longmapsto [\gamma_{s,t}] \oplus [\lambda_2]$

$[\mu] \longmapsto [\gamma_{p,q}] \oplus [\mu_2]$



$[\mu]_T \longmapsto [\mu]_V = 0$

6.8 (3)

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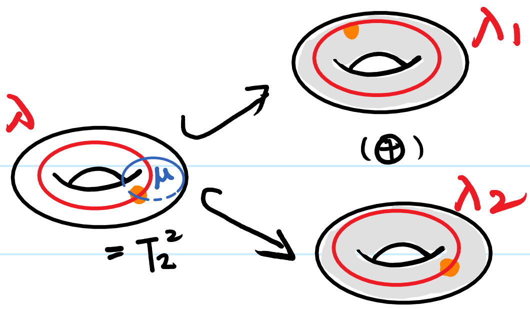

By 6.7

$\Longleftrightarrow$ $h_1 : \tilde{H}_1(X_1 \cap X_2) \longrightarrow \tilde{H}_1(X_1) \oplus \tilde{H}_1(X_2)$

$\mathbb{Z}[\lambda] + \mathbb{Z}[\mu] \qquad \mathbb{Z}[\lambda_1] \oplus \mathbb{Z}[\lambda_2]$

$$[\lambda] \longmapsto (s[\lambda_1] + t\,\cancel{[\mu_1]}^{\,0}) \oplus [\lambda_2] = s[\lambda_1] \oplus [\lambda_2]$$

$$[\mu] \longmapsto (p[\lambda_1] + q\,\cancel{[\mu_1]}^{\,0}) \oplus \cancel{[\mu_2]}^{\,0} = p[\lambda_1] \oplus 0$$

$\Rightarrow$ $h_1 : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} [\lambda] \\ [\mu] \end{pmatrix} \longmapsto \begin{pmatrix} s & 1 \\ p & 0 \end{pmatrix} \begin{pmatrix} [\lambda_1] \\ [\lambda_2] \end{pmatrix}$$

so $h_1 \cong \begin{pmatrix} s & 1 \\ p & 0 \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$, In fact, $\begin{pmatrix} 0 & 1 \\ 1 & -s \end{pmatrix} \begin{pmatrix} s & 1 \\ p & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$

and $\begin{pmatrix} s & 1 \\ p & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & -s \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ are invertible matrices over $\mathbb{Z}$.

$\Rightarrow$ $\ker h_1 \cong \ker \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = 0$

$\operatorname{Im} h_1 \cong \operatorname{Im} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \mathbb{Z} \oplus p\mathbb{Z}$

From (*):

$$\tilde{H}_2(L(p,q)) \cong \operatorname{Im} \partial_2 = \ker h_1 = 0$$

$$\tilde{H}_1(L(p,q)) \cong \tilde{H}_1(X_1) \oplus \tilde{H}_1(X_2) / \ker(h_1) = \mathbb{Z} \oplus \mathbb{Z} / \operatorname{Im}(h_1). = \mathbb{Z}/p\mathbb{Z}$$

Conclusion:

$$\boxed{H_n(L(p,q)) = \begin{cases} \mathbb{Z} & (n = 3 \text{ or } 0) \\ \mathbb{Z}/p\mathbb{Z} & (n = 1) \\ 0 & \text{Otherwise} \end{cases}}$$

$\square$