Rewrite Chapter 16: Dirichlet's Theorem and the Continuation of $\zeta(s)$ and $L(s,\chi)$ to $\mathbb{C}$

Instructor: *Chan Ieong Kuan*

Haoyu Zhu

Contents

1 Set-ups

1.1 How dense is a set of primes in all primes?

It is well-known that

$$\zeta(s) = \sum_{n\in\mathbb{N}_{>0}} \frac{1}{n^{-s}} = \prod_p (1-p^{-s})^{-1}.$$

The last equation holds for $s > 1$. Roughly speaking, $\zeta(s)$ has a simple pole at 1 with residue 1 since $\lim_{s\to 1^+}(s-1)\zeta(s) = 1$. (This is not rigorous because now we can only prove the case where $s \to 1^+$ through $s > 1$ rather than arbitrary direction on $\mathbb{C}$. Actually, one of our main goals is to continue $\zeta(s)$ to $\mathbb{C}$.) Therefore, as $s \to 1$, $\frac{\ln\zeta(s)}{\ln(s-1)^{-1}} \to 1$. In addition, as $s \to 1^+$, $\ln(s-1)^{-1} \to +\infty$ while $R(s) \triangleq \ln\zeta(s) - \sum_p p^{-s}$ remains bounded. (Details can be found in *GTM84*.) We conclude that,

Fact 1. $\lim_{s\to 1} \frac{\sum_p p^{-s}}{\ln(s-1)^{-1}} = \lim_{s\to 1} \frac{\ln\zeta(s)}{\ln(s-1)^{-1}} = 1$, where the sum runs over all positive primes.

Noticeably, for some set of positive primes $\mathscr{P}$ over which runs the sum, write

$$d(\mathscr{P}) = \lim_{s\to 1} \frac{\sum_{p\in\mathscr{P}} p^{-s}}{\ln(s-1)^{-1}}.$$

When $\mathscr{P}$ is finite, $d(\mathscr{P})$ reduces to 0, and when it consists of all but finitely many positive primes, $d(\mathscr{P})$ remains 1. In addition, when there are two disjoint $\mathscr{P}_1, \mathscr{P}_2$, then $d(\mathscr{P}_1 \cup \mathscr{P}_2) = d(\mathscr{P}_1) + d(\mathscr{P}_2)$. These properties inspire us to measure the density for all set of positive primes.

Definition 1. A set of positive primes $\mathscr{P}$ is said to have *Dirichlet density* if $\lim_{s\to 1} \frac{\sum_{p\in\mathscr{P}} p^{-s}}{\ln(s-1)^{-1}}$ exists, defined as the Dirichlet density. Magically, this definition coincides the natural definition of density, $\lim_{x\to\infty} \frac{|\{\mathscr{P}\cap\mathbb{N}_{\le x}\}|}{|\{\text{all primes in } \mathbb{N}_{\le x}\}|}$, once they both exist.

1.2 Dirichlet Theorem

Theorem 1. Suppose $a, m \in \mathbb{Z}$, with $(a, m) = 1$. Let $\mathscr{P}(a; m) = \{p \text{ positive primes} \mid p \equiv a \pmod m\}$. Then $d(\mathscr{P}(a; m)) = 1/\phi(m)$.

Remark 1. Obviously, Theorem1 infers that $|\mathscr{P}(a; m)| = \infty$. It also shows that the "quantity" of $\mathscr{P}(a; m)$ for fixed m is almost the same no matter given which a, so $\mathscr{P}(a; m)$'s devide all positive primes almost equally. In fact, when we sum up both side over $\{a \mid (a, m) = 1\}$, we obtain $1 = 1$. Notice that for finite $x \in \mathbb{N}$, $\pi_{a,m}(x) \triangleq |\mathscr{P}(a; m) \cap \mathbb{N}_{\le x}|$ might differ for different a. The difference is called *Chebyshev's bias*. I visulize the bias $\pi_{1,4}(x) - \pi_{3,4}(x)$ for $0 \le x \le 10^8$ by MatLab, see figure 1, and it seems like $\pi_{3,4}(x)$ is usually larger than $\pi_{1,4}(x)$.

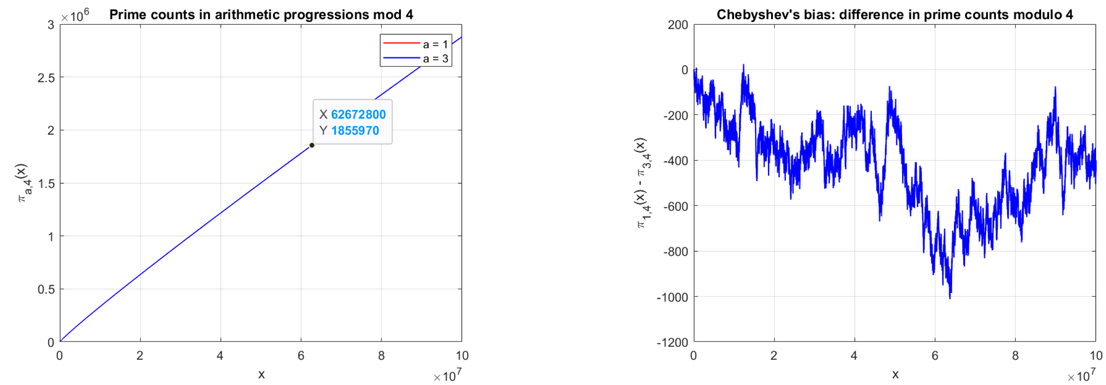


Figure 1: Chebyshev's bias

1.3 Dirichlet Characters and Dirichlet L-functions $L(s, \chi)$

Definition 2. We can generalize characters $\chi' : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*$ $(m > 0)$ to $\chi : \mathbb{Z} \to \mathbb{C}^*$ (which is called *Dirichlet Character modulo* m) in this way:

1. if $(n, m) = 1$, then $\chi(n) = \chi'(n)$.
2. otherwise, $\chi(n) = 0$.

More generally speaking, given any finite abelian multiplicative group A of order n, we deduce the Abelian group of characters $\hat{A} = \{\chi : A \to \mathbb{C}^*\}$ with identity $\chi_0 : a \mapsto 1$ for all $a \in A$, and multiplication is defined by $\phi\psi(a) = \phi(a)\psi(a)$. Values of χ are n^{th} roots of unity since χ's are homomorphisms.

The structures of A and $\hat{A}$ follow from the *theorem of the structure of finite Abelian groups*, and one can show that $A \cong \hat{A}$. Writing A and $\hat{A}$ as direct products of cyclic groups helps us prove the *Orthogonal Property of Characters* in $\hat{A}$. (Details are omited.)

Proposition 1 (Orthogonal Property of Characters)**.** Let A be a finite abelian group of order n and $\chi, \psi \in \hat{A}, a, b, \in A$. Then

1. $\langle \chi, \psi \rangle = \frac{1}{n} \sum_{a \in A} \chi(a)\overline{\psi(b)} = \delta(\chi, \psi)$.
2. $\langle a, b \rangle = \frac{1}{n} \sum_{\chi \in \hat{A}} \chi(a)\overline{\chi(b)} = \delta(a, b)$.

Let $A = (\mathbb{Z}/m\mathbb{Z})^*$ of order $\phi(m)$ and we generalize $\hat{A}$ to Dirichlet Character modulo m. Now $\chi_0(n) = \mathbf{1}_{(n,m)=1}(n)$. There are exacly $\phi(m)$ such characters. Set $\chi, \psi : \mathbb{Z} \to \mathbb{C}^*$ and $a, b \in \mathbb{Z}$, then

1. $\langle \chi, \psi \rangle = \frac{1}{\phi(m)} \sum_{a=0}^{m-1} \chi(a)\overline{\psi(b)} = \delta(\chi, \psi)$.

2. $\langle a, b\rangle = \frac{1}{\phi(m)} \sum_\chi \chi(a)\overline{\chi(b)} = \delta(a, b)$.

Definition 3 ($\ln L(s,\chi)$)**.** We can find $G(s,\chi)$ such that $\exp G(s,\chi) = L(s,\chi)$ for $s > 1$, though $\ln L(s,\chi)$ does not exist generally. In fact,

$$G(s,\chi) = \sum_p \sum_{k=1}^{\infty} (\frac{1}{k})\chi(p^k)p^{-ks}.$$

Fact 2.

$$\lim_{s\to 1} \frac{G(s,\chi_0)}{\ln \frac{1}{s-1}} = 1, \ \sum_\chi \overline{\chi(a)} G(s,\chi) = \phi(m) \sum_{p\equiv a(m)} p^{-s} + R_{\chi,a}(s),$$

where $(a, m) = 1$ and $R_{\chi,a}(s)$ remains bounded as $s \to 1$.

Proof. $\square$

Now to prove theorem 1 is equivalent to prove $\lim_{s\to 1} \frac{\sum_\chi \overline{\chi(a)} G(s,\chi)}{\ln \frac{1}{s-1}} = 1$, which is contributed mainly by χ_0, i.e. $\lim_{s\to 1} \frac{\overline{\chi_0(a)} G(s,\chi_0)}{\ln \frac{1}{s-1}} = \lim_{s\to 1} \frac{1\cdot G(s,\chi_0)}{\ln \frac{1}{s-1}} = 1$ according to Fact 2. So we only need to prove that $G(s,\chi)$ remains bounded as $s \to 1$ when $\chi \neq \chi_0$, then their summation devided by $\ln(s-1)^{-1}$ is negligible when $s \to 1$.

2 Continuation of $\zeta(s)$ and $L(s,\chi)$ to $\{Re(s) > 0\}$

Observe that $|a^s| = a^{Re(s)}$. Thus the series defining $\zeta(s)$ and $L(s,\chi)$ converges absolutely on $\{s \in \mathbb{C} \mid Re(s) > 1\}$, hence well-defined analytic functions there.

2.1 $\zeta(s)$

Lemma 1 (Abel's summation formula)**.** Suppose $\{a_n\}, \{b_n\}$ are sequences of complex numbers such that $\sum_{n=1}^\infty a_n b_n$ converges. Define partial sum $A_n = \sum_{i=1}^n a_i$ and suppose $A_n b_n \to 0$ as $n \to \infty$, then

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} A_n(b_n - b_{n+1}).$$

Fact 3. $\zeta(s) - (s-1)^{-1}$ can be continued to an analytic function on the region $\{s \in \mathbb{C} \mid Re(s) > 0\}$, so $\zeta(s)$ can be continued to the same region with the only simple pole at $s = 1$.

Proof. By the lemma, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \cdot 1 = \sum_{n=1}^{\infty} n(n^{-s} - (n+1)^{-s})$. Notice that $x \in (n, n+1)$, $n = [x], n^{-s} - (n+1)^{-s} = s\int_n^{n+1} x^{-s-1}\,dx$. Therefore, $\zeta(s) = \frac{s}{s-1} - s\int_n^{n+1} \langle x\rangle x^{-s-1}\,dx$, and for all x with $|\langle x\rangle| < 1$ the last integral converges, $\zeta(s) - (s-1)^{-1}$ is analytic on $\{s \in \mathbb{C} \mid Re(s) > 0\}$. □

2.2 $L(s, \chi)$

Lemma 2. Let χ be a nontrivial character modulo m. For all $N > 0$, we have

$$|\sum_{n=0}^{N} \chi(n)| \leq \phi(m).$$

Proof. $\chi(n)$ has a period of m and its value is always m^{th} root of unity. When $\chi \neq \chi_0$, $\sum_{a=0}^{m-1} \chi(a) = \langle \chi, \chi_0 \rangle = 0$. Therefore, $|\sum_{n=0}^{N} \chi(n)| \leq \sum_{n=0}^{m-1} |\chi(n)| = \phi(m)$. □

Theorem 2. Let χ be a nontrivial Dirichlet character modulo m. Then, $L(s, \chi)$ can be continued to an analytic function in the region $\{s \in \mathbb{C} \mid Re(s) > 0\}$. More importantly, $L(1, \chi)$ is a **nonzero** well-defined complex number.

Proof. □

Example 1 ($\chi(a) = (\frac{a}{p})$)**.** Set p a prime, χ is the Legendre symbol. We know Gauss sum $g(\chi) \triangleq \sum_{x=1}^{p-1} \chi(x)\zeta^x$, $\zeta = e^{2\pi i/p}$. Now we show that for $n, r \in \mathbb{Z}_p^*$,

$$\prod_{\chi(n)=-1} (1 - \zeta^n) \prod_{\chi(r)=1} (1 - \zeta^r)^{-1} = \exp(g(\chi)L(1, \chi)).$$

We rewrite the left-hand side as a product over all $x \in \mathbb{Z}_p^*$:

$$\prod_{\chi(n)=-1} (1 - \zeta^n) \prod_{\chi(r)=1} (1 - \zeta^r)^{-1} = \prod_{x=1}^{p-1} (1 - \zeta^x)^{-\chi(x)}.$$

Taking logarithms defined by expanding $\log(1 - \zeta^x) = -\sum_{k\geq 1} \frac{\zeta^{xk}}{k}$ yields

$$\log(\text{LHS}) = -\sum_{x=1}^{p-1} \chi(x) \log(1 - \zeta^x) = \sum_{x=1}^{p-1} \chi(x) \sum_{k=1}^{\infty} \frac{\zeta^{xk}}{k} = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{x=1}^{p-1} \chi(x)\zeta^{xk}.$$

For a prime p and the Legendre symbol χ, the Gauss sum satisfies

$$\sum_{x=1}^{p-1} \chi(x)\zeta^{xk} = \begin{cases} \chi(k)g(\chi), & p \nmid k, \\ 0, & p \mid k. \end{cases}$$

Hence

$$\log(\text{LHS}) = \sum_{\substack{k=1 \\ p\nmid k}}^{\infty} \frac{\chi(k)g(\chi)}{k} = g(\chi) \sum_{\substack{n=1 \\ p\nmid n}}^{\infty} \frac{\chi(n)}{n} = g(\chi)L(1,\chi).$$

Exponentiating gives LHS $= \exp\big(g(\chi)L(1,\chi)\big)$. We apply this identity to the simplest cases.

Case $p = 3$**.** Here $\chi(1) = 1,\ \chi(2) = -1$ and $\zeta = e^{2\pi i/3}$. Then

$$\text{LHS} = \frac{1-\zeta^2}{1-\zeta} = e^{i\pi/3}, g(\chi) = i\sqrt{p} = i\sqrt{3}.$$

Hence

$$\exp\big(i\sqrt{3}\, L(1,\chi)\big) = e^{i\pi/3} \implies i\sqrt{3}\, L(1,\chi) = i\pi/3 + 2\pi i k,\ k \in \mathbb{Z}.$$

Since $L(1,\chi)$ is a positive real number, we obtain $L(1,\chi) = \dfrac{\pi}{3\sqrt{3}}$.

Case $p = 5$**.** Here $\chi(1) = \chi(4) = 1,\ \chi(2) = \chi(3) = -1$ and $\zeta = e^{2\pi i/5}$. Then

$$\text{LHS} = \frac{(1-\zeta^2)(1-\zeta^3)}{(1-\zeta)(1-\zeta^4)} = \frac{3+\sqrt{5}}{2}, g(\chi) = \sqrt{5}.$$

Thus

$$\exp\big(\sqrt{5}\, L(1,\chi)\big) = \frac{3+\sqrt{5}}{2} \implies L(1,\chi) = \frac{1}{\sqrt{5}} \log \frac{3+\sqrt{5}}{2} = \frac{2}{\sqrt{5}} \log \frac{1+\sqrt{5}}{2}.$$

2.3 Proof of Theorem1

Proof. Set χ a nontrivial Dirichlet character. We want to show $G(s,\chi)$ remains bounded as $s \to 1$. Roughly speaking, when $s \to 1$, $G(s,\chi) \nrightarrow \pm\infty$ as a "logarithm" $\Leftrightarrow$ $L(s,\chi) \nrightarrow +\infty$ or 0, which is exactly what Theorem 2 told us. Now we prove it in rigorous language. We sometimes write $L(s,\chi) = L_\chi(s)$ to emphase that it is a function of s. Recall that we have already analytically extended $L(s,\chi)$ to $\{Re(s) > 0\}$, see Theorem 2.

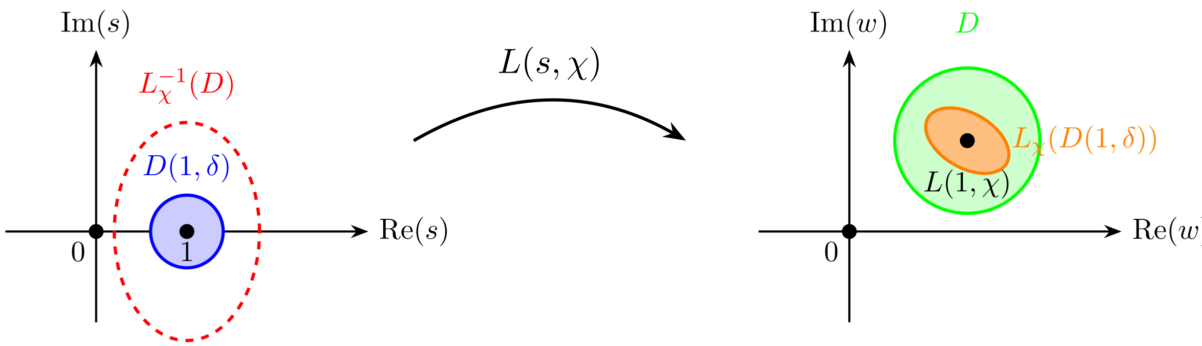
Since $L(1,\chi) \neq 0$, we can find a sufficiently small neighborhood

$$L(s,\chi) = L_\chi(s) : D(1,\delta) \to D,$$

with $\delta > 0, D$ a disc centered at $L(1,\chi)$, such that the branch point $0 \notin D$. Now on D we can define a single-valued branch of logarithm $Ln(z)$, and $G(s,\chi) = Ln(L(s,\chi)) + 2\pi i N$ with some fixed integer N for $s \in D(1,\delta)$. This implies

$$\lim_{s\to 1} G(s,\chi) = Ln(\lim_{s\to 1} L(s,\chi)) + 2\pi i N = Ln(L(1,\chi)) + 2\pi i N,$$

which is clearly bounded. $\square$



3 Functional equations and Continuation to $\mathbb{C}$ using $\Gamma(\frac{s}{2})$

3.1 Preparations

3.1.1 Calculations with Fourier Transformation

Lemma 3 (Poisson summation)**.** Define $\hat{f}(x) = \int_{-\infty}^{+\infty} f(t)e^{-2\pi ixt}\,dt$.

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e(nx).$$

When $x = 0$, we obtain $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$.

Proof. □

Lemma 4 (FT of $e^{-\pi x^2}$ and $xe^{-\pi x^2}$)**.** Let $f(x) = e^{-\pi x^2}$ and $g(x) = xe^{-\pi x^2}$, then $\hat{f}(x) = e^{-\pi x^2}$ and $\hat{g}(x) = -ixe^{-\pi x^2}$.

Proof. □

3.1.2 Gamma function

Definition 4 (Gamma function)**.** For $Re(s) > 0$, the Gamma function is defined by

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,dt.$$

It extends meromorphically to $\mathbb{C}$ with simple poles at $s = -k$ ($k = 0, 1, 2, \ldots$).

Proposition 2. Here are some important properties of the Gamma function:

1. **Functional equation:**
$$\Gamma(s+1) = s\Gamma(s), \qquad \Gamma(1) = 1.$$
Hence for positive integers n: $\Gamma(n) = (n-1)!$.

2. **Reflection formula:**
$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$
In particular, $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(1/2+n) = \frac{(2n)!}{4^n n!}\sqrt{\pi}$.

3. **Residues at poles:** For $k = 0, 1, 2, \ldots$,
$$\operatorname*{Res}_{s=-k} \Gamma(s) = \lim_{s\to -k}(s+k)\Gamma(s) = \lim_{s\to -k} \frac{\Gamma(s+k-1)}{s(s+1)\ldots(s+k-1)} = \frac{(-1)^k}{k!}.$$
In particular, $\Gamma(0)$ is a simple pole with residue 1.

4. **Euler's product representation:**
$$\Gamma(s) = \frac{1}{s}\prod_{n=1}^{\infty} \frac{(1+1/n)^s}{1+s/n} = \lim_{n\to\infty} \frac{n!n^s}{s(s+1)...(s+n)}.$$

5. **Weierstrass product:**
$$\frac{1}{\Gamma(s)} = se^{\gamma s}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-s/n},$$
where $\gamma = \lim_{n\to+\infty} H_n - ln(n)$,the Euler–Mascheroni constant. The (tail of the) infinite product converges absolutely on $\mathbb{C}$ and Weierstrass form is valid on entire $\mathbb{C}$, too.

3.2 Functional equation of $\zeta(s)$

Theorem 3. Define $\zeta^*(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$. $\zeta^*(s)$ can be meromorphically continued to $\mathbb{C}$, hence $\zeta(s)$. In fact, $\zeta(s)$ has a unique simple pole at 1. What's more, $\zeta^*(s) = \zeta^*(1-s)$, so
$$\zeta(s) = \frac{\pi^{\frac{s-1}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)}{\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})} = \pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\zeta(1-s). \tag{1}$$

Remark 2. We will show in the proof that $\zeta^*(s)$ only has two simple poles on $\mathbb{C}$, one at $s = 0$ provided by $\Gamma(s/2)$, and the other at $s = 1$ provided by $\zeta(s)$. Therefore, by virtue of the simple poles of $\Gamma(s/2)$, we know that negative even integers are trivial zeros of $\zeta(s)$. Remarkably, the symmetry $\zeta^*(s) = \zeta^*(1-s)$ demonstrate the reason why Riemmann is interested in the line $\{s \in \mathbb{C} \mid Re(s) = 1/2\}$ on which, as he hypothized, lie all nontrivial zeros of $\zeta(s)$.

Proof. Suppose $Re(s) > 1$,

$$\begin{aligned}
\zeta^*(s) &= \pi^{-s/2} \int_0^\infty e^{-t} t^{s/2} \sum_{n\geq 1} \frac{1}{n^s} \frac{dt}{t} && \text{(plug-in)} \\
&= \pi^{-s/2} \sum_{n\geq 1} \int_0^\infty e^{-n^2 t} t^{s/2} \frac{dt}{t} && (t \mapsto tn^2 \text{ in each summand}) \\
&= \sum_{n\geq 1} \int_0^\infty e^{-\pi n^2 t} t^{s/2} \frac{dt}{t} && (t \mapsto t\pi) \\
&= \int_0^\infty \frac{1}{2} \left(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} - 1 \right) t^{s/2} \frac{dt}{t} && (\text{extend the summation to } \mathbb{Z}) \\
&= A + \int_0^1 \frac{1}{2} \left(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} - 1 \right) t^{s/2} \frac{dt}{t} && \left(A \triangleq \int_1^\infty \frac{1}{2} \left(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} - 1 \right) t^{s/2} \frac{dt}{t} \right) \\
&= A + \int_0^1 \frac{1}{2} \left(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} \right) t^{s/2} \frac{dt}{t} - \frac{1}{s} && \left(\frac{1}{2} \int_0^1 t^{\frac{s}{2}} \frac{dt}{t} = \frac{1}{s} \right) \\
&\triangleq A + B - 1/s && (\text{denote the second integral by } B)
\end{aligned}$$

We calculate the second part, write B:

$$\begin{aligned}
B &= \int_0^1 \frac{1}{2} \left(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} \right) t^{s/2} \frac{dt}{t} \\
&= \int_0^1 \frac{1}{2} \left(\frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}} e^{-\pi n^2 / t} \right) t^{s/2} \frac{dt}{t} && \text{(Poisson lemma)} \\
&= \int_1^\infty \frac{1}{2} \left(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} \right) t^{\frac{1-s}{2}} \frac{dt}{t} && (t \mapsto 1/t) \\
&= \int_1^\infty \frac{1}{2} \left(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} - 1 \right) t^{\frac{1-s}{2}} \frac{dt}{t} - \frac{1}{1-s} && \left(\frac{1}{2} \int_1^\infty t^{\frac{1-s}{2}} \frac{dt}{t} = \frac{1}{s-1} \right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\zeta^*(s) &= \int_1^\infty \frac{1}{2} \left(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} - 1 \right) t^{s/2} \frac{dt}{t} + \int_1^\infty \frac{1}{2} \left(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} - 1 \right) t^{\frac{1-s}{2}} \frac{dt}{t} - \frac{1}{1-s} - \frac{1}{s} \\
&= \int_1^\infty \frac{1}{2} \left(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} - 1 \right) \left(t^{\frac{s}{2}} + t^{\frac{1-s}{2}} \right) \frac{dt}{t} - \frac{1}{1-s} - \frac{1}{s}
\end{aligned}$$

All integrals converge for all $s \in \mathbb{C}$, so $\zeta^*(s)$ is defined on $\mathbb{C} \setminus \{0, 1\}$, and $0, 1$ are simple poles. It is obvious that $\zeta^*(s) = \zeta^*(1-s)$. $\zeta(s) = \frac{\zeta^*(s)\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})}$ has a unique simple pole at $s = 1$ after continuation. $\square$

There are many popular memes about $\zeta(0), \zeta(-1)$, which show how funny it might be if we misunderstand $\zeta(s)$ at nonpositive integers as a summation of integers literally.

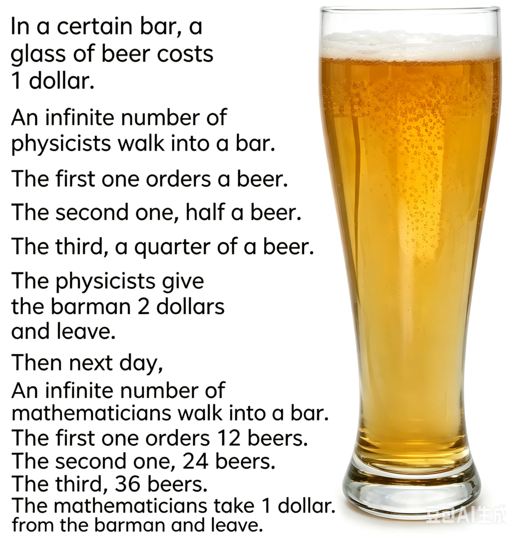


Figure 2: in a bar

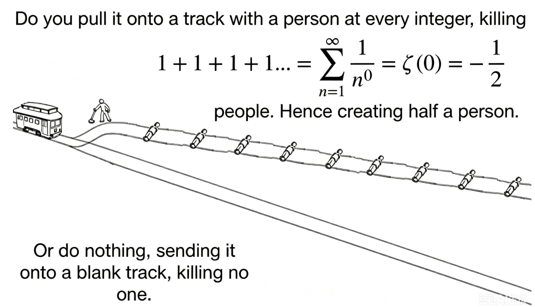


Figure 3: trolley problem

Now we will show the correct calculation of these two famous $\zeta(s)$.

Corollary 1. $\zeta(0) = -1/2, \zeta(-1) = -1/12$.

Proof. Calculate the residue at $s = 1$ for both sides in equation 1, then $1 = \pi^{1/2} \frac{\zeta(0)}{\Gamma(1/2)} \mathrm{Res}\{\Gamma(s), 0\}$, therefore $\zeta(0) = -1/2$. In addition, take $s = -1$, then $\zeta(-1) = \pi^{-3/2} \frac{\Gamma(1)}{\Gamma(-1/2)} \zeta(2) = -\frac{1}{12}$. $\square$

Remark 3 ($\zeta(s)$ at integers)**.** Define B_m the *Bernoulli numbers* by

$$B_0 = 1, \ (m+1)B_m = -\sum_{k=0}^{m-1} \binom{m+1}{k} B_k.$$

For $k \in \mathbb{Z}_{\geq 1}$, we now know the values of $\zeta(0) = -1/2, \zeta(-1) = -1/12, \zeta(-2k) = 0$, and $\zeta(1)$ does not exist. In the next section, we will calculate that

$$\zeta(1-k) = -\frac{B_k}{k}, \zeta(2k) = (-1)^{k+1} \frac{B_{2k}(2\pi)^{2k}}{2(2k)!}.$$

Unluckily, however, values of $\zeta(2k+1)$ are still open problems. See 4.1 for details.

3.3 Functional equation of $L(s,\chi)$

Theorem 4 (even character)**.** Let χ be an even primitive character modulo $m > 1$, i.e. period of χ is exactly m and $\chi(-1) = 1$. Define $L^*(s,\chi) = \frac{m}{\pi}^{\frac{s}{2}}\Gamma(\frac{s}{2})L(s,\chi)$. $L^*(s,\chi)$ can be continued to $\mathbb{C}$ **without poles**, hence $L(s,\chi)$. In fact, $\zeta^*(s) = \frac{g(\chi)}{\sqrt{m}}\zeta^*(1-s)$,

Proof. □

Theorem 5 (odd character)**.** Let χ be an odd primitive character modulo $m > 1$ ($\chi(-1) = -1$.) Define $L^*(s,\chi) = \frac{m}{\pi}^{\frac{s+1}{2}}\Gamma(\frac{s+1}{2})L(s,\chi)$. $L^*(s,\chi)$ can be continued to $\mathbb{C}$ **without poles**, hence $L(s,\chi)$. In fact, $\zeta^*(s) = \frac{-ig(\chi)}{\sqrt{m}}\zeta^*(1-s,\bar{\chi})$.

Proof. □

4 Another way to extend $\zeta(s)$ and $L(s,\chi)$ to $\mathbb{C}$ using $\Gamma(s)$

4.1 Bernoulli numbers and Formulas for $\sum_{k=0}^{n-1} k^m$

The motivation of *Bernoulli numbers* is quite simple. For $n \geq 1, S_m(n) \triangleq \sum_{k=0}^{n-1} k^m$, it is well-known that

$$\begin{aligned} S_0(n) &= 0^0 + 1^0 + 2^0 + \ldots + (n-1)^0 = n \\ S_1(n) &= 0^1 + 1^1 + 2^1 + \ldots + (n-1)^1 = \frac{n(n-1)}{2} \\ S_2(n) &= 0^2 + 1^2 + 2^2 + \ldots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6} \end{aligned}$$

We want to deduce a formula like the above for all $S_m(n)$. We observe that

$$(k+1)^{m+1} - k^{m+1} = 1 + \sum_{j=1}^{m} \binom{m+1}{j} k^j.$$

Let k run from 1 to $(n-1)$ and we sum them up to obtain

$$n^{m+1} - 1 = (n-1) + \sum_{j=1}^{m} \binom{m+1}{j} S_j(n) \tag{2}$$

$$\Leftrightarrow (m+1)S_m(n) = n^{m+1} - n - \sum_{j=1}^{m-1} \binom{m+1}{j} S_j(n). \tag{3}$$

Denote the coefficients of term n in $S_m(n)$ by B_m, then from the equation (3), we give an inductive definition of *Bernoulli numbers* as in Remark 3:

$$B_0 = 1,\ (m+1)B_m = -\sum_{k=0}^{m-1}\binom{m+1}{k}B_k \left(\Leftrightarrow \sum_{k=0}^{m}\binom{m+1}{k}B_k = 0\right), m \geq 1.$$

Here are some special values of Bernoulli numbers:

$$B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, B_5 = 0, B_6 = 1/42, B_7 = 0, \ldots$$

Question: does $B_{2k+1} = 0$ hold for all $k = 3, 5, 7, \ldots$? Yes!

Lemma 5.

$$\frac{t}{e^t - 1} = \sum_{m\geq 0} B_m \frac{t^m}{m!}.$$

Notice that $\frac{t}{e^t-1} - 1 + \frac{t}{2} = \sum_{m\geq 2} B_m \frac{t^m}{m!}$ is an even function, so $B_{2k+1} = 0$ for $k \in \mathbb{Z}_{\geq 1}$.

Proof. We aim to show $t = (e^t - 1)\sum_{m\geq 0} B_m \frac{t^m}{m!}$.

$$\begin{aligned} RHS &= (e^t - 1)\sum_{m\geq 0} B_m \frac{t^m}{m!} \\ &= \sum_{n\geq 1}\frac{t^n}{n!}\sum_{m\geq 0} B_m \frac{t^m}{m!} \\ &= \sum_{k\geq 1}\left(\sum_{\substack{n+m=k\\ n\geq 1\\ m\geq 0}} \frac{B_m}{n!m!}\right) t^k, \end{aligned}$$

where

$$\sum_{\substack{n+m=k\\ n\geq 1\\ m\geq 0}} \frac{B_m}{n!m!} = \frac{1}{k!}\sum_{m=0}^{k-1}\binom{k}{m}B_m = \begin{cases} 1 & \text{when } k = 1, \\ 0 & \text{otherwise, by definition of } B_m. \end{cases}$$

Therefore, $RHS = t$. □

Theorem 6 (Formula for $S_m(n)$)**.**

$$(m+1)S_m(n) = \sum_{k=0}^{m}\binom{m+1}{k}B_k n^{m+1-k}.$$

Proof. Consider $f(t) = \frac{e^{nt}-1}{e^t-1}$. On one hand,

$$f(t) = \sum_{r=0}^{n-1} e^{rt} = \sum_{r=0}^{n-1}\sum_{m\geq 0} \frac{(rt)!}{m!} = n + \sum_{m\geq 1} \frac{S_m(n)}{m!} t^m.$$

On the other hand,

$$\begin{aligned} f(t) &= \frac{t}{e^t-1} \cdot \frac{e^{nt}-1}{t} \\ &= \sum_{m\geq 0} \left(\sum_{\substack{k+h-1=m \\ k\geq 0 \\ h\geq 1}} \frac{B_k n^h}{k!h!} \right) t^m. \\ &= \sum_{m\geq 0} \left(\frac{1}{(m+1)!} \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m+1-k} \right) t^m. \end{aligned}$$

We complete the proof by comparing the coefficients of t^m. □

Definition 5 (Bernoulli polynomials)**.**

$$\frac{te^{xt}}{e^t-1} \triangleq \sum_{n\geq 0} \frac{B_n(x)}{n!} t^n.$$

One can show that $B_n(x) = \sum_{m=0}^{m} B_m \binom{n}{m} x^{n-m}$, which is very useful in numerical analysis.

4.2 $\zeta(s)$ and its special values at integers

We consider $\Gamma(s)\zeta(s)$ and $2^{1-s}\Gamma(s)\zeta(s)$ first. Actually, by definition 4,

$$\Gamma(s)\zeta(s) \overset{(t\mapsto nt)}{=} \int_0^\infty \sum_{n\geq 1} e^{-nt} t^s \frac{dt}{t} = \int_0^\infty \frac{1}{e^t-1} t^{s-1}\, dt,$$

which converges as $t \to 0$ if and only if $s \in (1, +\infty)$, so does $2^{1-s}\Gamma(s)\zeta(s) = \int_0^\infty \frac{e^{-2t}}{1-e^{-2t}} t^s \frac{dt}{t}$. Neither of them help extend $\zeta(s)$. However, we fortunately find that $(1-2^{1-s})\Gamma(s)\zeta(s) = \int_0^\infty \left(\frac{e^{-t}}{1-e^{-t}} - \frac{e^{-2t}}{1-e^{-2t}} \right) t^s \frac{dt}{t} = \int_0^\infty \left(\frac{1}{1+e^t} \right) t^{s-1}\, dt$ converges as $t \to 0$ when $s > 0$ hence when $Re(s) > 0$.

We write, in this section, $\zeta^*(s) = (1-2^{1-s})\Gamma(s)\zeta(s)$. We will show that it can be extended to entire $\mathbb{C}$ plane.

Lemma 6. Define $R(x) = \frac{x}{1-x} - \frac{2x^2}{1-x^2} = \frac{x}{1+x}, R_0(t) = R(e^{-t}), R_m(t) = \frac{d^m}{dt^m}(R_0(t)) \triangleq e^{-t}P_m(e^{-t})(1+e^{-t})^{-2m}$, where P_m's are polynommials, $m \geq 1$. Then,

$$R_{k-1}(\infty) = 0, R_{k-1}(0) = \frac{B_k}{k}(1-2^k).$$

Sketch of proof.

$$R_0(t) = \frac{1}{t}\left(\frac{t}{e^t-1} - \frac{2t}{e^{2t}-1}.\right)$$

Use lemma 5 and then compare $R_{k-1}(0)$ and the coefficients of the Taylor expansion of $R_0(t)$ at $t = 0$. □

Theorem 7.

$$\zeta^*(s) = \frac{(-1)^k}{s(s+1)\ldots(s+k-1)}\int_0^\infty R_k(t)t^{s+k-1}\,dt,\ k \geq 1.$$

Equivalently,

$$(1-2^{1-s})\Gamma(s+k)\zeta(s) = (-1)^k\int_0^\infty R_k(t)t^{s+k-1}\,dt,\ k \geq 1. \tag{4}$$

Since k can be any sufficiently large integer, for any $s \in \mathbb{C}$, we can find some $k \geq 1$ such that the integral exists for s. Therefore, $\zeta^*(s)$ is extended to $\mathbb{C}$, hence $\zeta(s)$. Moreover, we can now calculate $\zeta(s)$ thanks to equation (4).

Proof. □

Finally, we will show the value of $\zeta(s)$ at integers. We first focus on the negatives.

Corollary 2. $\zeta(1-k) = (-1)^{k-1}\frac{B_k}{k}$ for $k \geq 1$.

Proof. Let $s = 1-k$, we obtain $(1-2^k)\zeta(1-k) = (-1)^{k-1}R_{k-1}(0)$. Use lemma 6. Notice that $\zeta(-2m) = 0$ for $B_{2m+1} = 0, m \geq 1$. □

In addition, we can infer the value of $\zeta(s)$ at even positive integers using the functional equation (1): $\zeta(s) = \pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\zeta(1-s)$. Be careful that this equation is useless at odd positive integers, where $\Gamma(\frac{1-s}{2})$ has a pole and $\zeta(1-s) = 0$. That's why $\zeta(2k+1)$ remains mysterious today.

Corollary 3. $\zeta(2k) = (-1)^{k+1}\frac{B_{2k}\,(2\pi)^{2k}}{2\,(2k)!}$, for $k \geq 1$.

Proof. Set $s = 2k$ with $k \geq 1$:

$$\zeta(2k) = \pi^{2k-\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2} - k\right)}{\Gamma(k)} \zeta(1-2k).$$

Known values at negative integers give $\zeta(1-2k) = (-1)^{2k-1}\frac{B_{2k}}{2k} = -\frac{B_{2k}}{2k}$. Now simplify $\Gamma\left(\frac{1}{2} - k\right)$. Using the reflection formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$ with $z = \frac{1}{2} + k$:

$$\Gamma\left(\frac{1}{2} + k\right)\Gamma\left(\frac{1}{2} - k\right) = \frac{\pi}{\sin \pi(\frac{1}{2} + k)} = \frac{\pi}{\cos(\pi k)} = (-1)^k \pi.$$

Hence $\Gamma\left(\frac{1}{2} - k\right) = \frac{(-1)^k \pi}{\Gamma\left(\frac{1}{2}+k\right)}$. Also $\Gamma\left(\frac{1}{2} + k\right) = \frac{(2k)!\sqrt{\pi}}{4^k k!}$ and $\Gamma(k) = (k-1)!$. Therefore

$$\frac{\Gamma\left(\frac{1}{2} - k\right)}{\Gamma(k)} = \frac{(-1)^k \pi}{\Gamma\left(\frac{1}{2} + k\right)\Gamma(k)} = \frac{(-1)^k \pi}{\frac{(2k)!\sqrt{\pi}}{4^k k!}(k-1)!} = \frac{(-1)^k 4^k k \sqrt{\pi}}{(2k)!}.$$

Substitute back:

$$\zeta(2k) = \pi^{2k-\frac{1}{2}} \cdot \frac{(-1)^k 4^k k \sqrt{\pi}}{(2k)!} \cdot \left(-\frac{B_{2k}}{2k}\right) = (-1)^{k+1} B_{2k} \frac{4^k \pi^{2k}}{2(2k)!} = (-1)^{k+1} \frac{B_{2k}(2\pi)^{2k}}{2(2k)!}.$$

□

4.3 $L(s, \chi)$

To handle $L(s, \chi)$ we proceed in exactly the same way as for $\zeta(s)$.