The Architecture of Symmetry: From Lie Algebras to Root Systems

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1 Introduction: The Quest for Classification

Our goal is to classify the **simple Lie algebras** over the complex numbers $\mathbb{C}$. The path can be summarized as follows:

$$\text{Lie Algebras} \longrightarrow \text{Root Space Decomposition} \longrightarrow \text{Root Systems} \longrightarrow \text{Classification (Dynkin Diagrams)}$$

This article will trace this path, providing an overview of the key definitions, theorems, and constructions, finally lead to the main theorem:

Theorem 1.1 (Classification of Simple Lie Algebras)**.** The complex simple Lie algebras are in one-to-one correspondence with the irreducible root systems. They are classified into four infinite families (the **classical algebras**) and five **exceptional algebras**.

Example 1.2. A_n: $\mathfrak{sl}(n+1,\mathbb{C})$.(Lie algebra, Cartan matrix, Root system, Dynkin graph.)

2 The Algebraic Foundation

2.1 Lie Algebras and Ideals

Definition 2.1. A **Lie algebra** $\mathfrak{g}$ is a vector space over a field (typically $\mathbb{C}$) equipped with a bilinear operation, the **Lie bracket** $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, satisfying:

1. **Antisymmetry:** $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$.
2. **Jacobi Identity:** $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

Definition 2.2. A subspace $\mathfrak{i} \subseteq \mathfrak{g}$ is an **ideal** if $[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}$, i.e., for all $x \in \mathfrak{g}$ and $y \in \mathfrak{i}$, we have $[x, y] \in \mathfrak{i}$.

Ideals are the Lie algebra analogue of normal subgroups. They allow us to define quotient algebras $\mathfrak{g}/\mathfrak{i}$.

2.2 Solvable and Nilpotent Algebras

We define two important chains of ideals to measure how "non-abelian" a Lie algebra is.

Definition 2.3. The **derived series** of $\mathfrak{g}$ is defined by:

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}], \quad \dots, \quad \mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}].$$

$\mathfrak{g}$ is **solvable** if $\mathfrak{g}^{(n)} = 0$ for some n.

Definition 2.4. The **lower central series** of $\mathfrak{g}$ is defined by:

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}^1], \quad \dots, \quad \mathfrak{g}^n = [\mathfrak{g}, \mathfrak{g}^{n-1}].$$

$\mathfrak{g}$ is **nilpotent** if $\mathfrak{g}^n = 0$ for some n.

Remark 2.5. Nilpotency is a stronger condition than solvability. Every nilpotent Lie algebra is solvable, but not vice versa.

$$\text{Abelian} \hookrightarrow \text{Nilpotent} \hookrightarrow \text{Solvable}$$

A key example: the Lie algebra of strictly upper-triangular matrices is nilpotent, while the algebra of all upper-triangular matrices is solvable.

2.3 Simple and Semisimple Lie Algebras

Definition 2.6. A Lie algebra $\mathfrak{g}$ is **simple** if it is non-abelian and has no non-trivial ideals.

Simple Lie algebras are the irreducible building blocks. We now define a broader class that is built from these blocks.

Definition 2.7. The **radical** of $\mathfrak{g}$, denoted $\mathrm{rad}(\mathfrak{g})$, is the unique maximal solvable ideal.

Definition 2.8. A Lie algebra $\mathfrak{g}$ is **semisimple** if $\mathrm{rad}(\mathfrak{g}) = 0$.

Example 2.9. Show that the following lie algebras are semisimple:

- A_n: $\mathfrak{sl}(n+1, \mathbb{C})$
- B_n: $\mathfrak{so}(2n+1, \mathbb{C})$
- C_n: $\mathfrak{sp}(2n, \mathbb{C})$
- D_n: $\mathfrak{so}(2n, \mathbb{C})$

See [link], Theorem 11.2.1 in page 51, for reference.

The following theorem provides several equivalent characterizations of semisimplicity, each offering a different perspective.

Theorem 2.10 (Equivalent Definitions of Semisimple)**.** For a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, the following are equivalent:

1. **(Canonical)** $\mathrm{rad}(\mathfrak{g}) = 0$. ($\Leftrightarrow$ No nontrivial Abelian ideals.)

2. **(Structural)** $\mathfrak{g} = I_1 \oplus I_2 \oplus \ldots \oplus I_k$, where each I_i is a simple ideal.

3. **(Technical)** The Killing form is non-degenerate.

What is the Killing form?

Definition 2.11. The **Killing form** $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is defined by:

$$\kappa(x, y) = \operatorname{tr}(\operatorname{ad}_x \circ \operatorname{ad}_y),$$

where $\operatorname{ad}_x(y) = [x, y]$ is the adjoint representation. The Killing form is a symmetric bilinear form. (But it is not necessarily an inner product, i.e. not necessarily positive definite.)

3 The Structure of Semisimple Algebras

3.1 The Root Space Decomposition

To understand the structure of a semisimple Lie algebra $\mathfrak{g}$, we want to simultaneously diagonalize as many operators as possible. We do this by choosing a special subalgebra.

Definition 3.1. A **maximal toral subalgebra** (or **Cartan subalgebra**) $\mathfrak{h}$ is a maximal abelian subalgebra such that for all $h \in \mathfrak{h}$, the operator ad_h is semisimple (diagonalizable).

Since the operators $\{\operatorname{ad}_h : h \in \mathfrak{h}\}$ commute and are semisimple, they can be simultaneously diagonalized. This leads to the fundamental decomposition of $\mathfrak{g}$.

Theorem 3.2 (Root Space Decomposition). Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$. Then,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where for $\alpha \in \mathfrak{h}^*$ (the dual space of $\mathfrak{h}$), we define the **root space**

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

The set $\Phi \subset \mathfrak{h}^*$ is the finite set of **roots** for which $\mathfrak{g}_\alpha \neq 0$.

Remark 3.3.
- $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \mid [h, x] = 0 \ \forall h \in \mathfrak{h}\} = \mathfrak{g}_0$, the 0-eigenspace.
- $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Phi$.
- $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ (with the convention $\mathfrak{g}_\gamma = 0$ if $\gamma \notin \Phi \cup \{0\}$).
- There exists a unique $t_\alpha \in \mathfrak{h}$ such that $\alpha(h) = \kappa(t_\alpha, h)$ for all $h \in \mathfrak{h}$. We can define an **inner product** $(\cdot, \cdot) : \Phi \times \Phi \to \mathbb{C}$ by $(\alpha, \beta) = \kappa(t_\alpha, t_\beta)$. Why is it an inner product? Show the positive definite property since all t_α's live in $\mathfrak{h}$.

Proposition 3.4 (Orthogonal Properties). See Humphrey GTM9, page 37 (proposition 8.3)

Proposition 3.5 (Integrality Properties). See Humphrey GTM9, page 39 (proposition 8.4)

4 The Geometric Turn: Abstract Root Systems

The root set Φ is not an arbitrary collection of vectors. It possesses a highly rigid and symmetric geometry. We now abstract its essential properties.

4.1 Axioms of a Root System

Let E be a finite-dimensional Euclidean space with inner product $(\cdot,\cdot)$.

Definition 4.1. A subset Φ of E is a **root system** if it satisfies:

1. Φ is finite, spans E, and does not contain 0.
2. If $\alpha \in \Phi$, the only multiples of α in Φ are α and $-\alpha$.
3. Φ is closed under reflection through the hyperplane perpendicular to any root: for all $\alpha, \beta \in \Phi$,
$$s_\alpha(\beta) := \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi.$$
4. **Integrality Condition:** The number $\langle \beta, \alpha\rangle := 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$ is an integer for all $\alpha, \beta \in \Phi$.

The integers $\langle \beta, \alpha\rangle$ are called the **Cartan integers**. Given the inner product, we naturally define the dual system $\Phi^\vee = \{\text{coroots}\}$ by:
$$\Phi^\vee = \{\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)} \mid \alpha \in \Phi\}.$$

Definition 4.2. The **Weyl group** W is the finite group generated by the reflections $\{s_\alpha \mid \alpha \in \Phi\}$. In fact, it is a subgroup of $Aut(\Phi)$.

Definition 4.3 (Rank)**.** $dim(E)$ is called the rank of the root systems. Try to find some rank-1 or rank-2 root systems!

4.2 Simple Roots and the Cartan Matrix

To classify root systems, we choose a convenient basis.

Definition 4.4. A subset $\Delta = \{\alpha_1, \dots, \alpha_r\} \subset \Phi$ is a set of **simple roots** if it is a basis for E and every root $\beta \in \Phi$ can be written as $\beta = \sum k_i\alpha_i$ where all $k_i \in \mathbb{Z}_{\geq 0}$ or all $k_i \in \mathbb{Z}_{\leq 0}$.

The simple roots allow us to define a compact, integral matrix that encodes the entire geometry of the root system.

Definition 4.5. The **Cartan matrix** A of the root system Φ with respect to the simple roots $\Delta = \{\alpha_1, \dots, \alpha_r\}$ is the $r \times r$ matrix with entries
$$A_{ij} = \langle \alpha_j, \alpha_i\rangle = 2\frac{(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)}.$$

The Cartan matrix is independent of the choice of simple roots up to reordering and it determines Φ up to isomorphism(GTM9, proposition 11.1). Its properties ($A_{ii} = 2$, $A_{ij} \leq 0$ for $i \neq j$, and $A_{ij} = 0 \iff A_{ji} = 0$) allow us to represent it by a Dynkin diagram. In conclusion, to classify all (semi)simple Lie algebras, we only need to classify the Dynkin diagrams, so we transform an algebraic problem into a combinatorics problem (relatively easier).

5 The Grand Classification

The fundamental theorem is that the classification of simple Lie algebras over $\mathbb{C}$ reduces to the classification of irreducible root systems. How can we classify the Dynkin Diagrams in terms of the Axioms? See GTM9, theorem 11.4 in page 70.

Theorem 5.1 (Classification of Simple Lie Algebras)**.** The complex simple Lie algebras are in one-to-one correspondence with the irreducible root systems. They are classified into four infinite families (the **classical algebras**) and five **exceptional algebras**.

5.1 The Classical Lie Algebras

Type	Lie Algebra	Description	Rank
A_n	$\mathfrak{sl}(n+1,\mathbb{C})$	Traceless $(n+1)\times(n+1)$ matrices	$n \geq 1$
B_n	$\mathfrak{so}(2n+1,\mathbb{C})$	Odd-dimensional orthogonal algebras	$n \geq 2$
C_n	$\mathfrak{sp}(2n,\mathbb{C})$	Symplectic algebras	$n \geq 3$
D_n	$\mathfrak{so}(2n,\mathbb{C})$	Even-dimensional orthogonal algebras	$n \geq 4$

5.2 The Exceptional Lie Algebras

Type	Dimension
G_2	14
F_4	52
E_6	78
E_7	133
E_8	248

5.3 Constructing Lie Algebras from a Cartan Matrix

We can define a Lie Algebra given the following data:

1. **Cartan Matrix**: An $n \times n$ matrix $A = (a_{ij})$ with integer entries satisfying:
   - $a_{ii} = 2$ for all i,
   - $a_{ij} \leq 0$ for $i \neq j$,
   - $a_{ij} = 0 \iff a_{ji} = 0$,
   - There exists a diagonal matrix D such that DA is symmetric and positive definite. (Generalized Cartan Matrix does not require positive definite property.)
2. **Generators**: A set of $3n$ generators:

$$\{e_1, \ldots, e_n, \quad f_1, \ldots, f_n, \quad h_1, \ldots, h_n\}.$$

   The h_i span a Cartan subalgebra $\mathfrak{h}$.

The Lie algebra $\mathfrak{g}(A)$ of rank n is the quotient of the free Lie algebra on the generators by the following relations:

- **Cartan Relations:**
$$[h_i, h_j] = 0 \quad \text{for all } i, j.$$
- **Standard Relations:**
$$\begin{aligned} [h_i, e_j] &= a_{ij} e_j, \\ [h_i, f_j] &= -a_{ij} f_j, \\ [e_i, f_j] &= \delta_{ij} h_i. \end{aligned}$$
- **Serre Relations (for $i \neq j$):**
$$\begin{aligned} (\operatorname{ad} e_i)^{1-a_{ij}}(e_j) &= 0, \\ (\operatorname{ad} f_i)^{1-a_{ij}}(f_j) &= 0, \end{aligned}$$
  where $(\operatorname{ad} x)(y) = [x, y]$.

Triangular Decomposition

The resulting algebra has a *triangular decomposition*:

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

where $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) is the subalgebra generated by the e_i (resp. f_i).

Example 5.2 ($\mathfrak{sl}_3(\mathbb{C})$)**.** Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. The Serre relations for $(i, j) = (1, 2)$ and $(2, 1)$ are:

$$\begin{aligned} (\operatorname{ad} e_1)^2(e_2) = [e_1, [e_1, e_2]] = 0, \quad (\operatorname{ad} e_2)^2(e_1) = [e_2, [e_2, e_1]] = 0, \\ (\operatorname{ad} f_1)^2(f_2) = [f_1, [f_1, f_2]] = 0, \quad (\operatorname{ad} f_2)^2(f_1) = [f_2, [f_2, f_1]] = 0. \end{aligned}$$

These relations, together with the standard ones, define the 8-dimensional algebra $\mathfrak{sl}_3(\mathbb{C})$.